Immersion of Riemannian Geometries in Flat Geometries of Higher Dimension; Frames and Variables Adapted to Causal Slicing*

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1. Introduction

Hugo Wahlquist and I first met Jerzy Plebański in 1967 in New York during the second Texas Conference, We remember still his (second) question to us: "Tell me, how did you guys get the disease?)' Fascination with the physics and mathematics of general relativity is indeed a disease, a chronic affair, with which many of us feel privileged to have been afflicted. I am so very pleased to be at this symposium, to honor Professor Plebański's life of contributions and associations in our wonderful field, and to join with his colleagues and students in wishing him a most Happy Birthday.

I will discuss two problems of immersion, of (curved) Riemannian or pseudo-Riemannian manifolds seen as submanifolds of higher dimensional flat manifolds (Euclidean or pseudo-Euclidean.) The first is classic, that of two-dimensional spaces of constant negative curvature, immersed in ordinary Euclidean 3-space; by introducing intrinsic coordinates, arising from the immersion itself, the sine-Gordon equation and original transformation of Bäcklund are fount{. The second problem will be that of Ricci-flat 4-spaces, which are well known to be, locally at least, immersible in flat Euclidean spaces of ten dimensions. We will find the partial differential equations of both these immersions to explicitly show their causal property--the uniqueness of their integration from Cauchy-Kowaleski data set on one dimensional and three dimensional slices, respectively.

The partial differential equations of Riemannian immersion are invariantly formulated as exterior differential systems set on the orthogonal frame bundle over the immersing space. Since in these cases the immersing spaces arc 'flat, three and

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ten dimensional, respectively, the bundles are in fact group spaces, 1S0(3) and 1S0(10). 1S0(3) is six dimensional, the group of translations and rotations of flat 3-space E_3. 1S0(10) is 55 dimensional, the group of translations (10) and rotations (45) of E_{10}. The Cartan-Maurer structure equations of ISO(n) invariantly express its Lie algebraic structure in terms of n left (right)-invariant basis l-forms ω^μ and n(n-1)/2 basis l-forms $\omega^\mu_\nu (= -\omega^\nu_\mu$ in the case of positive definite signature):

$$d\omega^\mu + \omega^\mu_\nu \wedge \omega^\nu = 0$$
$$d\omega^\mu_\nu + \omega^\mu_\sigma \wedge \omega^\sigma_\nu = 0 \tag{1}$$

We will use this basis to set the differential ideals determining the immersions.

2. Cartan-Kähler Theory

Cartan-Kähler theory of (real, analytic) sets of partial differential equations considers closed exterior differential ideals on spaces of combined dependent and independent variables, dimension n. The generic maximal integral manifolds of an ideal—submanifolds on which, when pulled back, the ideal vanishes--- are the solutions. If these are g-dimensional, any g basis 1- forms which, pulled back or restricted to them, still remain linearly independent can be chosen as giving suitable independent variables; the n-g others then are denoted as dependent, satisfying the associated partial differential equations. We have briefly summarized C-K theory in several papers[1]; a recent excellent monograph by Yang[2] is highly recommended. Several diagnostic tests and techniques there described and justified will be of essential use in the following.

First, given an exterior differential ideal I, it is important to recognize its Cauchy characteristic vectors– vector fields which, when contracted with any form in I yield, again, a form in I. Their importance stems from the observation that all of them must lie in the maximal integral manifolds (otherwise an integral manifold of larger dimensionality could immediately be constructed!) Cauchy characteristic vectors thus give integral manifolds a fiber structure. The second essential diagnostic calculation is to find the set of Cartan (integer) characters s = $\{s_0, s_1, \ldots . s_{g-1}\}$. These are in principal found from the ranks of a nested sequence of linear homogeneous algebraic equations for the components of a set of vectors V_1, V_2, ... V_g that, from a generic point, can be integrated to span the integral manifold that is a g- dimensional solution. The genus g is in fact determined from the criteria that $s_0 + s_1 + \ldots + s_{g-1} \leq n - g$ while no further independent vector V_{g+1} exists. If the equality holds there are no arbitrary functions in the final construction---a well-set

Cauchy-Kowaleski integration—of a g-dimensional solution from a submanifold of lower dimension on which initial data are set, The last non-zero integer in the set s gives that dimension, and the number of initial data as functions on it. In the cases reported below the integers s have been calculated by Hugo Wahlquist using a Monte Carlo program to explore the ranks of the nested, and so interrelated, sets of ostensibly linear algebraic equations posed by Cartan's theory[1]. This method of finding s seems to be essential, as we must work in a large number of dimensions, and moreover there are 3-forms and 4-forms in the ideals. An explicit example is given in the Appendix.

The technique of prolongation leads to deep analyses of the algebraic structures underlying a set of partial differential equations, and to discovery of intrinsic, or adapted, coordinates. Roughly, prolongation is the addition of new forms to an ideal, and the simultaneous consistent introduction of new variables. Cartan used the term prolongation to mean introduction of higher partial derivatives-jet variables- -together with the equations relating them. We have used it also to mean systematic introduction of non-local variables---loosely, potentials and pseudo-potentials--- and shown how this can lead to discovery of inverse scattering solutions, Bäcklund transforms and (if the ideal I contains only l-forms and 2-forms) other solution methods based on Kit-Moody algebras. Prolongation also brings in the possibility of generalized invariance generators of the partial differential equations (or, as we have called them, isovectors of the ideals). This is all admirably discussed by Dan Finley and J. K. McIver in a paper now in press[3].

3. Ideals for Surfaces Immersed in E_3

The construction shown in Figure 1. illustrates and explains the "method of moving frames" approach to the classic surface immersion problem that is found in many elementary differential geometry texts, Changing notation to accord with these, we first rewrite the structure equations (1) for 1S(1(3), $\mu, \nu =$ 1,2,3, setting $\omega^1, \omega^2, \omega^3 \longrightarrow \theta^1, \theta^2, \theta^3$ and $\omega^1_2, \omega^2_3, \omega^3_1 \longrightarrow \omega^3, \omega^1, \omega^2$:

$$
\begin{aligned}
d\theta^1 - \omega^3 \wedge \theta^2 + \omega^2 \wedge \theta^3 &= 0 \\
d\theta^2 - \omega^1 \wedge \theta^3 + \omega^3 \wedge \theta^1 &= 0 \\
d\theta^3 - \omega^2 \wedge \theta^1 + \omega^1 \wedge \theta^2 &= 0 \\
d\omega^1 + \omega^2 \wedge \omega^3 &= 0 \\
d\omega^2 + \omega^3 \wedge \omega^1 &= 0 \\
d\omega^3 + \omega^1 \ \text{AU*} &= O
\end{aligned}
\tag{2}
$$

The immersion of a 2-surface is described by annulling a single l-form 6^3 and its closure, $d\theta^3$; that is, wc set the ideal I on 1S0(3) generated by

$$\begin{gathered} \theta^3 \\ \omega^1 \wedge \theta^2 - \omega^2 \wedge \theta^1 \end{gathered} \tag{3}$$

One intuitively thinks of orienting a moving orthogonal frame, or triad, at each point of the 2-surface, so that two frame vectors lie in the surface. This intuition is borne out by Cartan diagnostics of I: there is one Cauchy characteristic vector (dual to ω^3, which dots not appear in I), and $n = 6$, $s = \{1,1,1\}$, g = 3. A solution of I is thus a 3-dimensional sub-bundle of 1S0(3) with l-dimensional fibers over a 2-dimensional base. From the structure equations, it is immediately seen that this solution is an orthogonal frame bundle over two dimensions, with curvature 2-form $R^1_2 = -2\omega^1_3 \wedge \omega^3_2 = 2\omega^1 \wedge \omega^2$. A cross section of it is a particular orthogonal frame field, or (anholonomic) metric connection. According to the Cartan characters, each such bundle can be constructed by Cauchy-Kowaleski integration after giving one arbitrary function of position on a suitable 2-surface.

$$\begin{tikzcd} \begin{array}{c}\text{O(2) bundle}\\ \text{dim } 3\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\text{ISO(3)}\\ \text{dim } 6\end{array} \arrow[d] \\ \begin{array}{c}\text{M}\\ \text{dim } 2\end{array} \arrow[r] & \begin{array}{c}\mathbf{E_3}\\ \text{dim } 3\end{array} \end{tikzcd}$$

Figure 1. : Surfaces M immersed in E_3 have orthogonal frame bundles over them that are sub-bundles of 1S0(3).

The diagnostic alters dramatically if we specialize to surfaces of constant negative Gaussian curvature. We now consider the (closed) ideal I' generated by

$$*3$$

$$\omega^1 \wedge \theta^2 - \omega^2 \wedge \theta^1$$
$$\omega^1 \wedge \omega^2 + \theta^1 \wedge \theta^2 \tag{4}$$

Now we find $n = 6$, s = {1, 2, O} , $g = 3$, $s_2 =$ O! (This reduction of s_2 does not happen if we specialize to other classes of surfaces, e.g. constant positive curvature!) These constant curvature orthogonal frame bunches are uniquely constructed aft er giving two arbitrary funct icms cm a one dimensional submanifold (which must itself only annul θ^3). This can be called the discovery of a causal slicing of the 2- dimensional Riemannian geometry. Mathematically it allows us to go on and specialize to adapted cross sections— adapted frames-. by systematically searching for l-forms ζ such that $d\zeta$ = O, mod (1', ζ). Adding in such a l-form will only change s_0. In the present case two solutions exist; we use the same one as Chern and Terng[4], to "kill off' the $s_2 =$ O while still leaving a well-set ideal. That is, we finally consider an augmented ideal I" generated by

$$\theta^3$$
$$\omega^3 + \cot \tau \ w* + csc \ \tau \ \theta^2$$
$$\omega^1 \wedge \theta^2 - \omega^2 \wedge \theta^1$$
$$\omega^1 \wedge \omega^2 + \theta^1 \wedge \theta^2, \tag{5}$$

where τ is an arbitrary constant. Now $n = 6$, s = {2, 2}, $g = 2$.

We now can use both kinds of prolongation to find adapted coordinates. First, remember Cartan's Lemma, that if a set of l-forms ω_i^A satisfy ω_i^A A $\omega^i =$ O with the ω^i independent, then we may set ω_i^A -- $\phi_{ij}^A \omega^j$ = O, thereby introducing scalar fields ϕ_{ij}^A that must be symmetric on i and j. In the present case applying it to the 2-form in I", requiring θ^1 ancl θ^2 to be independent, introduces three scalars which the second 2-form in I" (for constant negative curvature) reduces to two. We get

$$\omega^1 + \frac{f-g}{f+g}\theta^1 + \frac{2}{f+g}\theta^2$$
$$\omega^2 + \frac{2 f_{f'g}}{f+g}\theta^1 \quad \frac{f-g}{f+- \ g}\theta^2 \tag{6}$$

Adding these to the ideal I" means we can drop the original two 2-forms, but at the price of adding in, for closure, the exterior derivatives of (6) (which simply follow from the structure relations). The last step is then to search for non-local prolongations, in this case just conservation laws, by setting, e.g.,

$$\sigma = A\omega^1 + B\omega^2 + C\theta^1 + D\theta^2 \tag{7}$$

with the coefficients functions of f and g, such that $d\sigma = 0$, mod I“. Solutions indeed are found:

$$\begin{aligned} &d\{\sqrt{1+f^2}\,(\theta^1 + \omega^1)\} \\ &d\{\sqrt{1-g^2}\,(\theta^1 - \omega^1)\} \end{aligned} \tag{8}$$

so we introduce “potential functions”, new scalar fields--coordinates-- x and t by setting

$$\begin{aligned} \sqrt{1+f^2}\,(\theta^1 + \omega^1) &= dx - dt \\ \sqrt{1+g^2}\,(\theta^1 - \omega^1) &= dx + dt \end{aligned} \tag{9}$$

One can then solve for all the original basis forms, ω^1, w*, W^3, θ^1, θ^2, explicitly in terms of the scalar coordinates f, g, x and t and their differentials. The partial differential equations that result from distinguishing f and g as dependent and x and t as independent variables, in the structure 2-form equations, are the famous pair of reciprocal sine-Gordon equations originally found by Bäcklund by laborious three dimensional geometrical construction:

$$\begin{aligned} \frac{\partial(\alpha+\psi)}{\partial x} &= a\sin(\alpha-\psi) \\ \frac{\partial(\alpha-\psi)}{\partial t} &= a^{-1}\sin(\alpha+\psi) \end{aligned} \tag{10}$$

where

$$\begin{aligned} a &= \cos\tau - \cot\tau \\ f &= \tan(\psi+\alpha) \\ g &= \tan(\psi-\alpha) \end{aligned} \tag{11}$$

4. **Ricci-flat** Four Spaces Immersed in E_{10}

We begin by dividing the basis forms in Eq. (1) into two sets, i, j., etc. = 1,2,3,4 and A, B, etc. = 5, 6, 7, 8, 9, 10. The structure equations then become

$$\begin{aligned} d\omega^i + \omega^i_k \wedge \omega^k + \omega^i_A \wedge \omega^A &= 0 \\ d\omega^A + \omega^A_B \wedge \omega^B + \omega^A_i \wedge \omega^i &= 0 \\ d\omega^i_j + \omega^i_k \wedge \omega^k_j + \omega^i_A \wedge \omega^A_j &= 0 \qquad \text{Gauss} \\ d\omega^A_B + \omega^A_k \wedge \omega^k_B + \omega^A_C \wedge \omega^C_B &= 0 \qquad \text{Ricci} \\ d\omega^i_A + \omega^i_k \wedge \omega^k_A + \omega^i_C \wedge \omega^C_A &= 0 \qquad \text{Codazzi} \end{aligned} \tag{12}$$

where we have indicated the famous names conventionally applied when these are pulled back to an immersed 4-manifold. The immersion is determined by the c1oscc1 exterior differential ideal I:

$$\begin{aligned} &\omega^{A} \\ &\omega_i^{A} \wedge \omega^i \end{aligned} \qquad (13)$$

There are 21 Cauchy characteristic vectors (since ω^i_j and ω^A_B are not explicitly in I), and s is calculated to be {6,6,6,6,6, O...}, $n = 55$, $g = 25$. We have 25 dimensional solutions, each a sub-bundle of 21 dimensional 0(4)@ $O(6)$ fibers over a 4-dimensional base.

A cross section of this so-called "Darboux " bundle yields not only an orthogonal frame field on **4-space**, but also 15 auxiliary $O(6)$ fields that arise from the immersion, These auxiliary $O(6)$ fields (or 0(5, 1), 0(4, 2) or 0(3, 3); wc have not had to be specific about signature) may prove to be as useful as those introduced in other formulations, for example, by complexifying orthogonal frame bundles. In terms of them, the Riemann tensor induced on the integral manifolds is quadratic. It may be significant that 0(4,2) is isomorphic to the conformal group CO(3, 1), and occurs in twistor analyses. Only in our first example, two dimensions immersed in three, did the Darboux bundle structure degenerate to that of an O(2) frame bundle, without auxiliary fields.

The most gratifying discovery Wahlquist and I have made is that, when we go on to consider Rice.i-flat and related geometries, Cartan character analysis shows the Darboux bundles to be "causally" determined from data on slices of lower dimension, here three. While this Cauchy property for general relativity is of course well known since work by Lichnerowicz, Choquet-Bruhat and ADM, here it emerges very elegantly and naturally, giving us conviction that the Darboux variables are especially well adapted to the algebraic structure of the field equations.

The Riemann curvature 2-forms can be read off from the Gauss structure equations in Eq.(12)

$$R^i_j = -2\omega^i_A \wedge \omega^A_j \qquad (14)$$

The Ricci tensor is coded in the 3-forms $R^i_j \wedge \omega^k \epsilon_{ijkl}$. The immersion ideal I' for Ricci-flat 4-geometries is

$$\begin{aligned} &\omega^{A} \\ &\omega_i^{A} \wedge \omega^i \\ &\omega^i_A \wedge \omega^j_A \wedge \omega^k\, \epsilon_{ijkl} \end{aligned} \qquad (15)$$

Cartan diagnostics of I' gives s = {6,6, 10,8, O...}, so solutions are determined by setting 8 functions on three dimensional slices. This is shown in Figure 2. Each such construction, of a bundle with 21 dimensional fibers over 4-space, is simultaneously 22 dimensions over three. Causality emerges together with an autonomous time variable. Similar analyses yield this causal structure for immersions of Einstein-Maxwell and Einstein-Klein -Gordon solutions, and for Ricci-flat three and five dimensional geometries (immersed in six and fifteen dimensional flat spaces, respectively[1]).

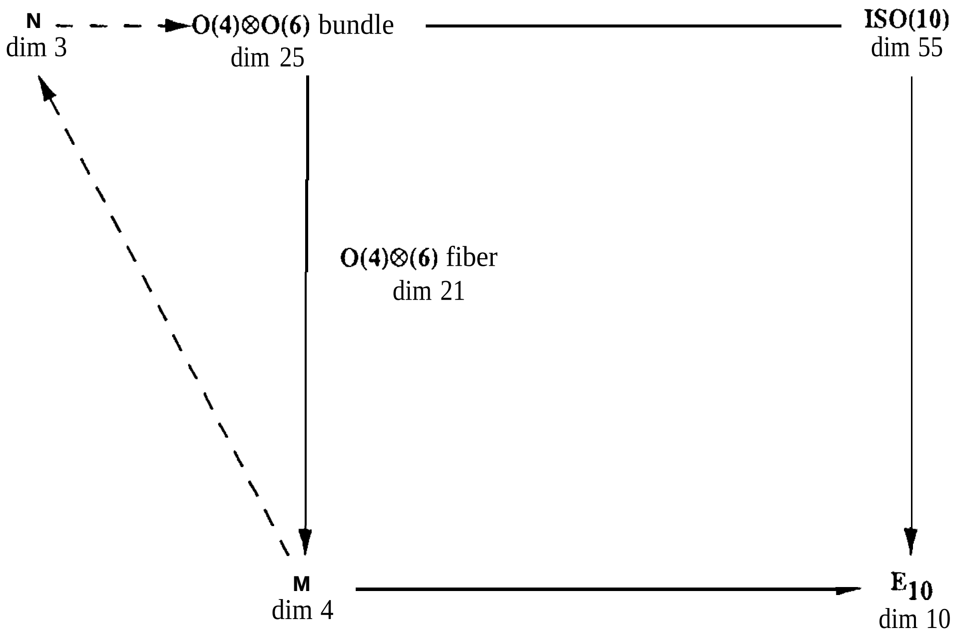

Figure 2. : Riemannian 4-geometries M immersed in E_{10} have Darboux frame bundles 0(4)@ 0(6) over them that are sub-bundles of 1S0(10). In the Ricci-flat case a C-K construction additionally gives M the structure of an 0(1) or line bundle over three dimensions (dashed arrows).

We are making progress in finding augmented ideals, with specialized frames, in a program analogous to that of Section 3 leading to adapted coordinates and the sine-Gordon equation. A first step has been to incorporate maximal slicing, a technique well known in numerical relativity. This proves to be nicely compatible with the causal structure already present. That is, we have recently analyzed the ideal I" generated by

$$\omega^{A}$$

$$\omega_i^{A} \wedge \omega^{i}$$

$$\omega_A^{i} \wedge \omega_A^{j} \wedge \omega^{k}\, \epsilon_{ijkl}$$

$$(\omega_4^1 \wedge \omega^1 + \omega_4^2 \wedge \omega^2 + \omega_4^3 \wedge \omega^3) \wedge \omega^4 \qquad (\Omega = 0)$$

$$\omega_4^1 \wedge \omega_4^2 \wedge \omega^1 \wedge \omega^2 + \omega_4^1 \wedge \omega_4^3 \wedge \omega^1 \wedge \omega^3 + \omega_4^2 \wedge \omega_4^3 \wedge \omega^2 \wedge \omega^3$$

$$(\omega_4^1 \wedge \omega^2 \wedge \omega^3 + \omega_4^2 \wedge \omega^3 \wedge \omega^1 + \omega_4^3 \wedge \omega^1 \wedge \omega^2) \wedge \omega^4 \qquad (\Theta = 0) \qquad (16)$$

and find that S_4 remains zero: s = $\{6, 6, 11, 10, 0, \ldots\}$, g = 22. This appears to be a neat demonstration of the (local) existence and uniqueness of maximal slicing. We now have $O(3) \otimes O(6)$ bundles over 4-space. Further specialization of the framing, and use of prolongation may yield explicit, adapted, variables and coordinates.

5. References

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Appendix

Monte Carlo calculation of Cartan characters: using the maximal-slicing, Ricci-flat ideal as an example

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The computation of Cartan characters used throughout this work follows very closely the exposition given in Estabrook and Wahlquist[1]. Here we describe the numerical method and display some typical results obtained for the maximal slicing ideal, Eq. (16).

The 55 basis l-forms of 1S0(10), $\{\omega^\mu, \omega^\mu_\nu\}$, (p, ν = 1,..., 10), are re-labeled into a l-dimensional array, $b_i\,(i = 1, \ldots 55)$, denoted Bi in the computer printout, Figure 3. The ordering of the basis forms is arbitrary, but can be (and here has been) chosen so as to optimize the computations. The set of forms for any of these immersion ideals can then be written uniformly as

$$1 - \text{forms} : \alpha^A = f_i^A b^i \qquad A = 1, \ldots, N_1$$

$$2 - \text{forms} : \beta^B = f_{ij}^B b^i \; \mathrm{A} \; b^j \qquad B = 1, \ldots, N_2$$

$$3 - \text{forms} : \gamma^C = f_{ijk}^C b^i \; \mathrm{A} \; b^j \; \mathrm{A} \; b^k \qquad C = 1, \ldots, N_3$$

$$\text{etc.} \qquad A(1)$$

where the coefficients f of every form are small integers. The ideal in Eq. (16) translates to the set of forms in Figure 4. The set of vectors spanning an integral element of the solution manifold is expressed as $V_\mathcal{K} = v^i_\mathcal{K} b_i$ where the subscript $\mathcal{K}$ labels the vectors in the set and b_i are the dual basis vectors satisfying $b_i \rfloor b^j = \delta^j_i$. Many such sets can be found at a point, and in principle, each would lead to a locally analytic solution manifold by Cauchy-Kowaleski integrations. We are here interested only in generic solutions for which the basis forms ω^μ (μ = 1,.2,3,4) of an immersed 4-space remain linearly independent. Consequently we demand that the first four vectors of a set must span (be non-degenerate on) this sub-space. If the ideal of forms allows such a set, the ideal is said by Cartan to be "involuntary" with respect to these ω^μ.

'MAXIMAL' BASIS 1 -FORMS

B1 = w5	B21 = W(3,7)	641 = W(5,10)
62 = w6	B22 = W(3,8)	B42 = W(6,7)
B3 = w?	B23 = W(3,9)	B43 = W(6,8)
B4 = w8	B24 = W(3,1 O)	B44 = W(6,9)
B5 = w9	B25 = W(4,5)	645 = W(6,1O)
B6=W1O	B26 = W(4,6)	B46 = W(7,8)
67 = W(1,5)	B27 = W(4,7)	B47 = W(7,9)
B8 = W(1,6)	B28 = W(4,8)	648 = W(7,1 O)
69= W(1,7)	629 = W(4,9)	B49 = W(8,9)
B10 = W(1,8)	630 = W(4,1 O)	B50 = w(8,1O)
B11 = W(1,9)	631 = W(1,4)	B51 = W(9,1O)
612 = W(1,10)	632 = W(2,4)	652 = w4
613 = W(2,5)	633 = W(3,4)	B53 = w3
B14 = W(2,6)	B34 = W(1,2)	654 = w2
B1S = W(2,7)	B35 = W(1,3)	B55 = w1
616 = W(2,8)	636 = W(2,3)	
B17 = W(2,9)	637 = W(5,6)	
B18 = W(2,1O)	638 = W(5,7)	
B19 = W(3,5)	639 = W(5,8)	
620 = W(3,6)	640 = W(5,9)	

Figure 3. : Labels of basis forms for the ideal *Maximal*.

The 1-forms α^A are initially ranked to determine the first Cartan character So, giving the number of independent 1-forms. If the α^A as given are independent, then so = $N1$. All ranking is accomplished by straightforward Gaussian elimination, so after ranking, the coefficient matrix f_i^A acquires upper triangular form; i.e., $f_i^A = 0$ for $i < A$. The actual calculations proceed using integer arithmetic to avoid any possible problems with numerical precision. The price paid for this is the rapid growth in the magnitudes of vector components, especially in large ideals including higher degree forms. The growth is minimized as far as possible by continually reducing over common factors, but this does not eliminate the problem for ideals as large as the maximal slicing 'ideal, even with 32-bit integers. The calculations often exceed this limit before a complete solution is reached. Having to repeat the calculation anew when this happens is not serious, however. Since each solution requires only a few seconds on a desktop computer, hundreds of complete solutions actually can be obtained in a couple of hours.

```
THIS IS THE IDEAL: 'MAXIMAL'

THE DIMENSION IS 55 .

THERE ARE 6 I-FORMS IN THE IDEAL:

# 1 = + B1

# 2 = + B2

# 3 = + B3

# 4 = + B4

# 5 = + B5

# 6 = + B6

THERE ARE 6 2-FORMS IN THE IDEAL:

# 1 = + B7^B55 +B13AB54+B19AB53 + B25^B52

# 2 = + B8^B55 + B14^B54 + B20AB53 + B26^B52

# 3 = + B9^B55 + B15ABS4 + B21^B53 + B27^B52

# 4 = + B1OAB55 +B16AB54 + B22^B53 + B28^B52

# 5 = + B11^B55 + B17^B54 + B23^B53 + B29^B52

# 6 = +B12AB55+ B18AB54+ B24AB53+B30AB52

THERE ARE 5 3-FORMS IN THE IDEAL:

# 1 = + B7^B13^B53 + B13^B19^B55 + B19^B7^B54 + B8^B14^B53 + B14^B20^B55 + B20^B8^B54
 + B9^B15^B53 + B15AB21 ^B55 + B21 ^B9^B54 + B1OAB16AB53 + B16AB22AB55 + B22^B10^B54
 +B1IAB17AB53 + 617AB23AB55 +B23AB11AB54 + B12AB18AB53 + B18^B24^B55 + B24^B12^B54

# 2 = + B7^B13^B52 + B13^B25^B55 + B25^B7^B54 + B8^B14^B52 + B14^B26^B55 + B26^B8^B54
 + B9AB15AB52 + B15AB27AB55 + B27^B9^B54 + B10^B16^B52 + B16A&@B55 + B28^B10^B54
 + B11^B17^B52 + 617AB29AB.55 + B29AB11AB54 + B12^B18^B52 + B18^B30^B55 + B30^B12^B54

# 3 = + B7^B19^B52 + B19^B25^B55 + B25^B7^B53 + B8^B20^B52 + B20^B26^B55 + B26^B8^B53
 + B9^B21 AB52 + B21 ^B27^B55 + B27AB9AB53 + B10AB22AB52 + B22^B28^B55 + B28.AB10AB53
 + B11^B23^B52 + B23AB29AB55 + B29AB11 AB53 + B12AB24AB52 + B24^B30^B55 +B30AB12AB53

# 4 = + B13^B19^B52 + B19^B25^B54 + B25^B13^B53 + B14^B20^B52 + B20^B26^B54 + B26^B14^B53
 + B15^B21 AB52 + B21^B27^B54 + B27AB15AB53 + B16AB22AB52 + B22^B28^B54 + B28AB16AB53
 + B17^B23^B52 + B23^B29^B54 + B29AB17AB53 + B18AB24AB52 + B24AB30ABS4 + B30AB18AB53

# 5 = + B31^B55^B52 + B32^B54^B52 + B33^B53^B52

THERE ARE 2 4-FORMS IN THE IDEAL:

# 1 = + B31^B54^B53^B52 - B32^B55^B53^B52 + B33^B55^B54^B52

# 2 = + B31^B32^B55^B54 + B31^B33^B55^B53 + B32^B33^B54^B53

THERE ARE O 5-FORMS IN THE IDEAL:
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Figure 4. : The ideal *Maximal* entered in the Monte Carlo program.

A first auxiliary vector $V_1 = v_1^i b_i$ is initially introduced with v_1^i being a set of randomly chosen (small!) integers. To be an integral element, some components of V_1 must then be adjusted to annul the 1-forms α^A

$$V_1 \rfloor \alpha^A = v_1^i f_i^A = 0 \qquad (A = 1, ..., s_0), \tag{A(2)}$$

a set of equations which is easily solved in reverse order taking advantage of the triangular structure of f_i^A. Any components of V_1 which are not solved for retain their randomly assigned values. A new set of 1-forms α_1^B is now generated by contracting V_1 on the 2-forms β^B

$$\alpha_1^B \equiv V_1 \rfloor \beta^B = v_1^i f_{[ij]}^B b^j. \tag{A(3)}$$

Here brackets around indices denote complete antisymmetrization, but. without the usual 1/n! factor. The full set of 1 -forms $\{\alpha^A, \alpha_1^B\}$ is now ranked to determine the second Cartan character (rank $= s_0 + s_1$).

A second auxiliary vector V_2, again taken with initially random integer components (except only for requiring linear independence from V_1), must annul the augmented set of 1-forms

$$V_2 \rfloor \alpha^A = v_2^i f_i^A = 0, \qquad V_2 \rfloor \alpha_1^B = v_2^j v_1^i f_{[ij]}^B = 0 \tag{A(4)}$$

which ensures that V_2 is a 1-dimensional integral element and V_1 and V_2 together span a 2-dimensional integral element. Again, after solving these equations any components of V_2 unconstrained by the equations retain their random integer values. Now V_2 is used to generate new 1-forms from the 2-forms

$$\alpha_2^B = V_2 \rfloor \beta^B = v_2^i f_{[ij]}^B b^j \tag{A(5)}$$

and V_1 and V_2 are jointly applied to the 3-forms to generate

$$\alpha_{12}^C = V_2 \rfloor V_1 \rfloor \gamma^C = v_2^j v_1^i f_{[ijk]}^C b^k. \tag{A(6)}$$

This entire set of 1-forms, $\alpha^A, \alpha_1^B, \alpha_2^B, \alpha_{12}^C$ is ranked to determine s_2(rank $= s_0 + s_1 + s_2$).

Proceeding in this way, each new initially random (but independent) auxiliary vector is required to annul all prior 1-forms and then contracted, together with all possible combinations of prior vectors, on the higher degree forms to generate new 1-forms which are then ranked. Finally, at some point no additional independent 1-forms will be obtained with, say, the k^{th} auxiliary vector V_k, so $s_k = 0$ and furthermore, all $s_l = 0$ $(l > k)$. If $k \leq g = n - \sum_{j=0}^{k-1} s_j$, the ideal is well-set and there are at least $(g - k)$ Cauchy characteristics.

Clearly, the random values of unconstrained components of the auxiliary vectors may lead to accidental degeneracy of the 1-forms generated from them at any step, so that the rank of the l-forms is not maximal. Such accidents, however, are relatively rare and are easily detected by repeating the solution several times; the maximal characters generally become readily apparent after a few repetitions. As mentioned before, repetitive solutions can be quickly accomplished.

The results of a short series of 8 solutions for the Cartan characters of the maximal slicing ideal are presented in Figure 5., and Figure 6. shows some of the corresponding sets of auxiliary vectors, which were carried through V_6. Here $k = 4$; i.e., $s_4 = 0$, and $g = 55 - 33 = 22$. Accordingly, the vector V_4 is completely determined by the first three auxiliary vectors, and the vectors V_5 and V_6 are Cauchy characteristics, of which there are 1S in all (note that the basis forms, B34 - B51, do not appear in the ideal).

'MAXIMAL' RESULTS

```
QM is approximately the largest integer encountered during a solution attempt.
IV is the number of involutory vectors ( if a desired number was specified at the start ).

1 QM = 5123392
  S={ 6 , 6 , 11 , 10 ,0 , 0 , O ,}      I V - 6
2 QM = 1.50427200
  S={ 6 , 6 , 11 , 10 ,0 , 0 , O ,}      IV= 6
3 QM = 4822910
  S-{ 6 , 6 , 1 1 , 1 0  ,0 , 0 , 0 ,}   IV = 6
4 QM = 78791376
  S={ 6 , 6 , 11 , 10 ,0 , 0 , O ,}      IV = 6
5 QM = 62895S225
  S={6,6, 11,1 O, O, O, O,}              IV = 6
6 QM = 4S630
  S-{6,6,11,10, 0,0, O ,}                IV = 6
7 QM = 274468500
  S={ 6 , 6 , 11,1 O, O, O, O,}          IV = 6
8 QM = 6748 S7656
  S={ 6 , 6 , 11 , 10 ,0 , 0 , O ,}      IV = 6

HIS IS ALL! TOTAL ATTEMPTS = 37
```

Figure 5. : Eight solutions found by the. Monte Carlo program.

NON-ZERO COMPONENTS OF VECTORS IN 'MAXIMAL' SOLUTION #1

V1: v7 --1 , v9 = 1 ,v10 - 1 , v11 - 1 , v13 --I , v15 = 1 , v16 = 1 , v17 --1
v19 = 1 , v20 = 1 , v21 = 1 , v22 --1 , v23 --1 , v24 = 1 , v25 - 1 , v26 = 1
v27 --1 , v29 = 1 , v32 = -2 , v33 --1 , v52 = 1 , v55 = 1

V2: v7 = -2 , v8 = 2 , v9 --1 , v1 I--1 , v12 --1 , v13 = 1 , v14 = 1 , v15 = -1
v16 = 1 , v17 = 1 , v18 --1 , v19 = 1 , v22 = 1 , v23 = 1 , v25 = 2 , v26 --1
v27 = 1 , v28 = 1 , v29 = 1 , v30 = 1 , v31 = 1 , v32 --1 , v52 = 1 , v54 = 1

V3: v7 = -9 , v8 = 9 , v9 = 3 , v1o --3 , v11 = -3 , v13 = -6 , v15 - 6 , v16 = 6
v17 = 3 , v19 - 17 , v20 = 30, v21 = 33 , v22 --3 , v23 = -3 , v25 = 15
v26 --3 , v27 = -3 , v29 = 3 , v30 = 3 , v31 = 9 , v32 = 3 , v33 = -3 , v52 = 3
VS3 - 3

V4: v7 = -1506 , v8 = 1311 , v9 = -862 ,v10 - 1902 , v1 I--1428 , v12 = 626
v13 --792 , v14 --117 , v15 = S66 , v16 = 2616 , v17 --1428 , v18 = 1340
v19 = 1350 , v20 --117 , v21 --862 , v22 = 1902 , v23 --1428 , v24 = 1340
v25 = 2220 , v26 --597 , v27 = 148 , v28 = -1902 , v29 = 2142 , v30 --626
v31 - 1190 , v32 --952 , v33 --1666 , v52 = 714

V5: v51 - 1

V6: v50 = 1

NON-ZERO COMPONENTS OF VECTORS IN ' MAXIMAL' SOLUTION #3

V1: v7 --1 , v8 = 1 ,v10 --1 , v1 I--1 , v12 --1 , v15 = 1 , v17 = 2 , v18 = 1
v19 = 1 , v20 = 1 , v21 --1 , v22 = -2 , v23 = 1 , v24 --1 , v25 --1 , v27 = 1
v28 = 3 , v29 --2 , v30 = 1 , v31 -'1 , v32 --1 , v33 = 1 , v52 = 1 , v55 = 1

V2: v7 = -2 , v8 = 1 , v9 = 1 , v10 - 2 , v11 --1 , v12 = 3 , v13 = 1 , v15 = 1
v17 --1 , v18 = 1 , v19 --1 , v20 = 1 , v21 --1 , v22 = 1 , v23 - 1 , v24 = 1
v25 - 1 , v26 --1 , v27 = 1 , v28 = 1 , v29 = 1 , v30 --1 , v31 --1 , v32 --1
v33 = 2 , v52 = 1 , v54 = 1

V3: V7 = 1 , v8 = -4 , v9 = 2 , v13 = 1 , v14 = -2 , v15 - 2 , v16 = -2 , v17 --6
v19 --36 , v20 --35 , v21 = 5 , v22 = -2 , v23 --2 , v24 = 2 , v25 --1 , v26 = 2
v27 --2 , v28 --2 , v29 = 2,v32--2, v33--2, v52 --2, v53 --2

V4: v7 = 2942 , v8 = -3500 , v9 --4436 ,v10 - 3154 , v11 --2920 , v12 = 45
v13 = 1694 , v14 --2876 , v15 --4436 , v16 = 4402 , v17 -4792 , v18 = 1293
v19 - 2006 , v20 --2876 , v21 - 4436 , v22 = 4402 , v23 - -3544 , v24 = 669
v25 --2318 , v26 = 3500 , v27 = 3812 , v28 = -5026 , v29 = 4168 , v30 --669
v33 --624 , v52 --624

V5: v51 --1

V6: v50 --1

Figure 6. : Explicit vector components of four solutions.

NON-ZERO COMPONENTS OF VECTORS IN ' MAXIMAL' SOLUTION #S

'1: v7 --1 , v8 --1 , v9 --I , v10 --I , vII- 1 , v12 --1 , v13 = 1 , v15 --I
v16 = -2 , v17 --1 , v18 --1 , v19 = -1 , v20 = 1 , v23 --1 , v24 = 1 , v25 = -1
v26 --1 , v27 --1 , V28 --1 , v29 = 1 , v30 = 1 , v32 = 1 , vS2- 1 , v55 = 1

'2: v7 = -1 , v8 = -2 , v9 --1 , V10--2, V12- 1 , v13 --1 , v14 = 1 , v15 = -1
v16 --1 , v18 --1 , v19 --1 , v20- 1 , v21 = 1 , v22 --1 , v24 --1 , v25 = 1
v26 - 1 , v27 --1 , v28 = -1 , V30 --1 , v31 --1 , v32 = 1 , v33 = 1 , v52 = 1
V54 - 1

'3: V7 --1 , v8 = 4 ,v10 --8 , v1 I--8 , v12 = 12, v13 = 7 , v14 = 12, v15 = 4
v16 = -12 , v17 --8 , v18 = -4 , v19 = -4 , v20 = 20, v21 = -19 , v22 --I, v23 = -4
V2S --7 , v26 -AI , v27 = -4 , v28 = 4 , v29 = 8,v30-4, v31--16, v32 = -4
V33 - 4 , v52 = 4 , v53 = 4

'4: v7 = 2761 , v8 = 2073 , v9 --17 ,v10 --147 , v11- 4262 , v12 = 2220
v13 = 2217 , v14 = 1S29 , v1 S--17 , v16 --147 , v17 = 4S34 , v18 = 2764
v19 = 296S , v20 = 2073 , v21 --17 , v22 --691 , v23 - 3990 , v24 = 2764
v25 --2489 , v26 --1801 , v27 - 289 , v28 - 419 , v29 - 4S34 , v30 --2492
V31 - S44 , V33 --544 , vS2 - -272

'S: v51 = 1

'6: v50 = 1

NON-ZERO COMPONENTS OF VECTORS IN ' MAXIMAL' SOLUTION #7

V1: v8 = 1 ,v10 --1 , v11 - 1 , v12 --1 , v13 --1 , v14 --1 , v15 - 1 , v16 = 1
v17 = -1 , v18 = 1 , v19 = 1 , v24 = 1 , v26 = 1 , v27 --1 , v28 --1 , v29 --2
V30 --1 , v32 = 1 , v33 --1 , v52 = 1 , v55 - 1

V2: v7 = -2 , v8 --1 ,v10 --1 , v1 I--3 , v12 = 1 , v13 = 1 , v14 = 1 , v15 - 1
v16 --1 , v17 --1 , v20 = 1 , v21 = 1 , v22 --1 , v23 --1 , v24 --1 , v25 - 1
v26 - 1 , v28 = 1 , v30 --1 , v31 = -2 , v32 = 1 , v33 --I ,vS2- 1 , v54 = 1

V3: v7 = 15 , v9 = 24, vii- 18, v12 = 6 , v13 = 15 , v14 = -6 , v15 - 12
v16 --6 , v17 = 12, v18 = 18, v19 = 23 , v20 = -24 , v21 = 16, v22 = 6
v23 = -6 , v24 = 6 , v25 --21 ,v26--6, v27--18, v28- 6 , v29 = -6 , v30 --6
v31 = 18, , v33 = -6 , v52 = -6 , v53 --6

V4V7 --2778 , v8 = 6682 , v9 --33684 ,v10 - 28188 , v11 --3715 , v12 = -9615
v13 --942 , v14 = 6682 , v1 S--31848 , v16 = 31860 , v17 -43 , v18 = -9615
v19 = 3648 , v20 - 6682 , v21 = -26340 , v22 - 28188 , v23 = 1793 , v24 --5943
v25 = 2778 , v26 -4846 , v27 = 31848 , v28 --30024 , v29 = 43 , v30 = 7779
v31 --1836 , v32 = 3672 , v33 - 1836 , v52 = 1836

V5: v51 --1

V6: vSO --1

Figure 6. : (Continued)